

**Particle Motion in a Yang-Mills Field:  
Wong's Equations and Spin- $\frac{1}{2}$  Analogues<sup>\*</sup>).**

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**Abstract**

*A complete, straightforward and natural Lagrangian description is given for the classical non-relativistic dynamics of a particle with colour or internal symmetry degrees of freedom moving in a background Yang-Mills field. This provides a new simple Lagrangian formalism for Wong's equations for spinless particles, and presents also their generalisation, in gauge covariant form, for spin- $\frac{1}{2}$  particles, within a complete Lagrangian formalism.*

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# 1 Introduction.

Wong's equations are the equations of motion in the non-relativistic mechanics of a spinless particle with colour or internal symmetry degrees of freedom moving in a background Yang-Mills field. The discussion originally [1] and subsequently has usually taken place within a Hamiltonian formalism. The geometrical context of Wong's equations has been widely discussed [2] [3] [4] [5] using several versions of such a formalism. But the long-standing problem regarding the difficulties of displaying a satisfactory Lagrangian formalism for the equations is still being explicitly remarked upon [6] at the present time. Papers written about the problem, and its analogue for particles of spin- $\frac{1}{2}$ , divide roughly into two subsets, according as whether or not Grassmann variables are used explicitly in the description of spin or colour. Such variables do indeed feature centrally in [7] [8] [9] [10] [11], the last source being a monograph, which reviews matters, provides collateral material and further references, as well as discussing the problem in terms of bosonic variables. Other references that also do so include [10] [12].

The present work actually differs in significant respects even from those of the cited papers that do use fermions to describe colour or spin. One main reason is that it does not use either a square root type of kinetic term in its starting Lagrangian, or else an equivalent form that employs an einbein. It is in fact non-relativistic from the outset, and any use that we make of the Dirac formalism to take account of constraints is trivial. Another aspect of our approach that gives it distinct characteristics is its use of Majorana fermions to describe spin and colour, using odd numbers of these where appropriate, and the logically related use of  $N = 1$  supersymmetry with an odd number (one!) of real Grassmann variables. The latter is important because the treatment first of the supersymmetric problem guides us towards suitable superactions, crucially using spinorial superfields to construct colour variables. This contrasts with the way that the fermionic description of spin arises from the use of scalar superfields. It also distinguishes our work from that of any previous paper that we have found.

Valuable background to the discussion of particle motion in a Yang-Mills field, including explicit formulas for sourceless fields of this type, and further references may be found in a recent interesting monograph [13].

In this paper, we exhibit a suitable Lagrangian for motion of a coloured particle in a Yang-Mills field and deduce from it the well-known Hamiltonian description of Wong's equations. Not only do we do this in the case of spinless particles, but we also give the corresponding formalism for particles of spin- $\frac{1}{2}$ , presenting therein in gauge covariant form a generalisation for spin- $\frac{1}{2}$  of Wong's equations.

We discuss the spin- $\frac{1}{2}$  case first, constructing a supersymmetric theory with scalar and spinor superfields. The former have the component expansions

$$\Phi_i = x_i + i\theta\phi_i \quad , \quad (1)$$

where the  $x_i$  are the particle position co-ordinates (in three spatial dimensions). Here  $\theta$  is the single real Grassmann number of an  $N = 1$  supersymmetric theory which possesses a single real supercharge. The  $\phi_i$  are Majorana fermion variables. The well known method [14] [15] [16] [17] [18] of building a spin vector  $S_i$  from them

$$S_i = -\frac{1}{2}i\varepsilon_{ijk}\phi_j\phi_k \quad (2)$$

guides us towards a suitable way of introducing colour variables. These come from the Majorana fermion variables  $\lambda_\alpha$  contained in the spinor superfields

$$\Lambda_\alpha = \lambda_\alpha + \theta F_\alpha \quad , \quad (3)$$

where  $\alpha = 1, 2, \dots, \dim g$ , where  $g$  is a compact Lie algebra, the Lie algebra of the colour or internal symmetry group of our theory. The  $F_\alpha$  feature as non-dynamical variables which we eliminate with the aid of their Euler-Lagrange equations, leaving behind as dynamical variables the  $\lambda_\alpha$  out of which we construct colour variables or charges  $j_\alpha$  as in (2) by means of the definition

$$j_\alpha = -\frac{1}{2}if_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma \quad . \quad (4)$$

The  $f_{\alpha\beta\gamma}$  here are the totally anti-symmetric structure constants of the Lie algebra  $g$ , and  $\lambda_\alpha$  and  $j_\alpha$  transform according to its adjoint representation.

It should be emphasised that, while the definition (4) is probably as well known as the corresponding definition of spin, (2), our use of spinorial superfields to bring the corresponding fermions,  $\lambda_\alpha$ , into the theory is clearly not.

We refer to a selection of papers [19] [20] [21] where the general principles of constructing a supersymmetric theory with a single Grassmann variable  $\theta$  out of superfields (1) and (3) are described. This leads naturally to a theory of a particle with position  $x_i$ , spin  $S_i$  and colour or charge  $j_\alpha$  moving in a background Yang-Mills field. We find an action with supersymmetry from which a complete canonical formalism emerges in a straightforward fashion. This yields also a Hamiltonian formalism and Hamilton equations for  $x_i$ ,  $S_i$  and  $j_\alpha$

that are the generalisation of Wong's equations to the case of particles of spin- $\frac{1}{2}$ . The fermion substructure of the spin and charge variables is hidden in this Hamiltonian formalism and the equations of motion it gives. But of course we now know it is nevertheless present and the key to the associated Lagrangian formalism. In fact, the supersymmetry is not essential to the discovery of the Lagrangian formalism. We have merely used it to guide us towards a suitable type of Lagrangian.

It follows that we can pass to the treatment of the case of spinless particles by omitting the  $\phi_i$  variables. Considerations to do with supersymmetry disappear at this point. But we are lead directly to a Lagrangian and a complete canonical formalism that contains the previously well-understood Hamiltonian discussion of Wong's equations. The Majorana fermions  $\lambda_\alpha$  of the theory are again hidden in the Hamiltonian formalism when this is finally couched in terms of the Hamiltonian and  $x_i$  and  $j_\alpha$ . And again we now know that the fermion substructure is lurking behind the scenes and essential to the Lagrangian formalism.

## 2 Supersymmetric Theory.

We consider the theory governed by the action

$$\begin{aligned} S &= \int dt d\theta \frac{1}{2} \left( i\dot{\Phi}_i D\Phi_i + \Lambda_\alpha D\Lambda_\alpha - gA_{i\alpha}(\Phi) f_{\alpha\beta\gamma} \Lambda_\beta \Lambda_\gamma D\Phi_i \right), \\ &= \int dt d\theta (K + \theta L) = \int dt L \quad . \end{aligned} \quad (5)$$

This involves the Lagrangian

$$\begin{aligned} L &= \frac{1}{2} \left( \dot{x}_i \dot{x}_i + i\dot{\phi}_i \dot{\phi}_i + i\dot{\lambda}_\alpha \dot{\lambda}_\alpha + F_\alpha F_\alpha + igA_{i\alpha} f_{\alpha\beta\gamma} \lambda_\beta \lambda_\gamma \dot{x}_i \right) \\ &+ \frac{1}{2} (g\phi_j \phi_i A_{i\alpha,j} f_{\alpha\beta\gamma} \lambda_\beta \lambda_\gamma + igA_{i\alpha} f_{\alpha\beta\gamma} (\lambda_\beta F_\gamma - F_\beta \lambda_\gamma)) \quad . \end{aligned} \quad (6)$$

The quantity  $K$  seen in (5) is used below in the construction of the supercharge  $Q$  by Noether's theorem. The variables  $F_\alpha$  do not appear in our Lagrangian as dynamical. Thus we can eliminate them using their Euler-Lagrange equation. This yields the result

$$\begin{aligned} L &= \frac{1}{2} \left( \dot{x}_i \dot{x}_i + i\dot{\phi}_i \dot{\phi}_i + i\dot{\lambda}_\alpha \dot{\lambda}_\alpha + igA_{i\alpha} f_{\alpha\beta\gamma} \lambda_\beta \lambda_\gamma \dot{x}_i \right) \\ &+ \frac{1}{4} F_{ij\alpha} f_{\alpha\beta\gamma} \lambda_\beta \lambda_\gamma \phi_i \phi_j, \end{aligned} \quad (7)$$

for  $L$  in terms of dynamical variables. Here

$$F_{ij\alpha} = \partial_i A_{j\alpha} - \partial_j A_{i\alpha} - g f_{\alpha\beta\gamma} A_{i\beta} A_{j\gamma} = \varepsilon_{ijk} B_{k\alpha} \quad , \quad (8)$$

defines the usual covariant field variables.

The provision of the canonical formalism is problem free. We find

$$\begin{aligned} p_i &= \dot{x}_i - gA_{i\alpha} j_\alpha \quad , \\ \{x_i, p_j\} &= \delta_{ij} \quad , \quad \{\lambda_\alpha, \phi_i\} = 0 \quad , \\ \{\phi_i, \phi_j\} &= -i\delta_{ij} \quad , \quad \{\lambda_\alpha, \lambda_\beta\} = -i\delta_{\alpha\beta} \quad , \end{aligned} \quad (9)$$

and the Hamiltonian

$$H = \frac{1}{2} (p_i + gA_{i\alpha} j_\alpha) (p_i + gA_{i\beta} j_\beta) + gB_{k\alpha} j_\alpha S_k \quad , \quad (10)$$

where the notations (2) and (4), still to be justified, have been used. Using the well-known transformation properties of our variables under supersymmetry transformations of Grassmann parameter  $\epsilon$ , we apply Noether's theorem in the form

$$-i\epsilon Q = \sum \delta X \frac{\partial L}{\partial \dot{X}} - i\epsilon K \quad , \quad (11)$$

where the sum is over all the dynamical variables  $X$ . This leads us to the expression for the supercharge

$$Q = (p_i + gA_{i\alpha} j_\alpha) \phi_i \quad . \quad (12)$$

Using the consequence

$$\{F, G\} = \frac{\partial F}{\partial x_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial x_k} + i(-)^f \frac{\partial F}{\partial \phi_k} \frac{\partial G}{\partial \phi_k} + i(-)^f \frac{\partial F}{\partial \lambda_\alpha} \frac{\partial G}{\partial \lambda_\alpha} \quad , \quad (13)$$

where  $f$  is the Grassmann parity of  $F$ , to calculate Poisson brackets, we can show that  $Q$  obeys the Poisson bracket relation

$$\{Q, Q\} = -2iH \quad . \quad (14)$$

This is the classical analogue of the more familiar quantal relationship  $Q^2 = H$ . We can also verify that  $Q$  generates canonically the supersymmetry transformation rules employed in the derivation of (12), a consistency check on the setting up of the formalism. We can also use (13) to show that

$$\begin{aligned} \{S_i, S_j\} &= \varepsilon_{ijk} S_k \quad , \\ \{j_\alpha, j_\beta\} &= f_{\alpha\beta\gamma} j_\gamma \quad , \end{aligned} \quad (15)$$

which justifies viewing  $S_i$  and  $j_\alpha$  as spin and colour or charge variables. In quantum theory in which we impose the anticommutation relations  $\phi_i \phi_j + \phi_j \phi_i = \delta_{ij}$ , we represent  $\phi_i$  (in the Schrödinger) picture in terms of Pauli matrices by  $\sigma_i/\sqrt{2}$ . It follows that we represent  $S_i$  by  $\sigma_i/2$ , indicating that we are dealing with the case of spin one-half.

We continue by using (13) to compute the time dependence of any classical variable  $Y$  via

$$\dot{Y} = \{Y, H\} \quad . \quad (16)$$

We treat the variables  $x_i, \dot{x}_i, \phi_i, S_i, \lambda_\alpha$  and  $j_\alpha$  in order. The first calculation gives back the definition of  $p_i$ . The others give rise to the equations of motion of the theory, the analogues for particles of spin one-half of Wong's equations, namely

$$\begin{aligned} \ddot{x}_i &= -g\varepsilon_{ijk} \dot{x}_j B_{k\alpha} j_\alpha - g D_i B_{j\alpha} j_\alpha S_j \quad , \\ \dot{S}_k &= g j_\alpha B_{i\alpha} \varepsilon_{ijk} S_j \quad , \\ D j_\alpha &= g B_{i\beta} S_i f_{\alpha\beta\gamma} j_\gamma \quad , \end{aligned} \quad (17)$$

where

$$\begin{aligned} D j_\alpha &= d j_\alpha / dt + g f_{\alpha\beta\gamma} \dot{x}_i A_{i\gamma} j_\beta \quad , \\ D_i B_{j\alpha} &= (\partial_i \delta_{\alpha\beta} + g f_{\alpha\beta\gamma} A_{i\gamma}) B_{j\beta} \quad . \end{aligned} \quad (18)$$

The two terms on the right-hand side of the first equation of (17) have a simple physical interpretation. The first is a non-Abelian generalisation of the Lorentz force and the second is a non-Abelian Stern-Gerlach force, coupling the spin to the gradient of the  $B$ -field.

It makes perfect sense also to consider in its own right a theory using the Hamiltonian specified in (10) in terms of variables  $x_i, S_i$  and  $j_\alpha$ , subject to the canonical relations  $\{x_i, p_j\} = \delta_{ij}$ , and (2) and (4). The same equations (17) follow as the Hamilton equations of this dynamical theory too of course. The fermionic substructure of spin and colour is hidden in this approach, and no Lagrangian formalism suggests itself naturally. This is indeed the situation that has obtained in the context of Wong's equations, i.e in the spinless case, for a long time.

### 3 Wong's Equations.

There is no obstacle at all to adopting the plan of simplifying the work of section two by discarding the fermionic variables  $\phi_i$  and ceasing to consider anything to do with spin or supersymmetry.

Thus we shall set out in the theory of spinless particles from the Lagrangian

$$L_0 = \frac{1}{2} \left( \dot{x}_i \dot{x}_i + i \lambda_\alpha \dot{\lambda}_\alpha + i g A_{i\alpha} f_{\alpha\beta\gamma} \lambda_\beta \lambda_\gamma \dot{x}_i \right) \quad . \quad (19)$$

We retain the definition of  $p_i$  contained in (9), and employ the canonical equations

$$\{x_i, p_j\} = \delta_{ij} \quad , \quad \{\lambda_\alpha, \lambda_\beta\} = -i \delta_{\alpha\beta} \quad . \quad (20)$$

We retain also the definition (4) of  $j_\alpha$ . Passage from  $L_0$  to the Hamiltonian  $H_0$  gives rise to the expression

$$H_0 = \frac{1}{2} (p_i + g A_{i\alpha} j_\alpha) (p_i + g A_{i\beta} j_\beta) \quad . \quad (21)$$

Calculating Hamilton's equations from (21) for the variables  $x_i, p_i, \lambda_\alpha$  and  $j_\alpha$  in turn, we get the equations of motion

$$\begin{aligned} \ddot{x}_i &= -g \dot{x}_j F_{ij\alpha} j_\alpha \quad , \\ D j_\alpha &= 0 \quad , \end{aligned} \quad (22)$$

where  $F_{ij\alpha}$  and  $D j_\alpha$  are as defined before in (8) and (18). The equations (22) agree exactly with Wong's equations, but here we have derived them within a complete canonical description of the classical mechanics of a spinless particle with colour or charge vector  $j_\alpha$  moving in a Yang-Mills field. It is complete in the sense that it stems in standard fashion from a suitable starting Lagrangian, (19). It makes a good check on the consistency of our work to verify that the equations of motion can also be derived as the Lagrange equations of (19).

We can also see easily that the historical description of Wong's equations is contained in our work. View  $H_0$  as a function of the (Grassmann even) variables  $x_i, p_i$  and  $j_\alpha$  subject only to the canonical equations

$$\{x_i, p_j\} = \delta_{ij} \quad , \quad \{j_\alpha, j_\beta\} = f_{\alpha\beta\gamma} j_\gamma \quad . \quad (23)$$

Hence, using

$$\{F, G\} = \frac{\partial F}{\partial x_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial x_k} + \frac{\partial F}{\partial j_\alpha} \frac{\partial G}{\partial j_\beta} f_{\alpha\beta\gamma} j_\gamma \quad , \quad (24)$$

to compute Poisson brackets of (Grassmann even) functions of the dynamical variables, we find, in this way too, Wong's equations and very easily; *e.g.*

$$d j_\alpha / dt = \{j_\alpha, H_0\} = \frac{\partial H_0}{\partial j_\beta} f_{\alpha\beta\gamma} j_\gamma = g \dot{x}_i A_{i\beta} f_{\alpha\beta\gamma} j_\gamma \quad . \quad (25)$$

Although this last discussion is very simple indeed, it gives no clue to the underlying fermionic nature of colour employed here to produce the sought after Lagrangian formalism.

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